Solving $R|\text{pmtn}|C_{max}$

Notation

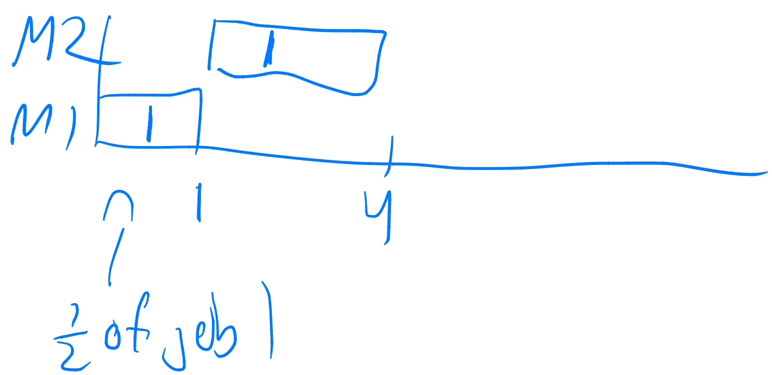
- n **jobs**
- m **machines**
- p_{ij} **is the processing time of job** j **on machine** i**.**

Interpretation:

In one unit of time, you can schedule $1/p_{ij}$ **of job** j **on machine** i**.**

$P_{11} = 2$

$P_{21} = 6$



$\frac{1}{2} + \frac{3}{6} = 1$

Step A: Assigning Jobs to Machines

We solve a linear program with variables:

$x_i j =$ **the time that job** jis **processed on machine** i

$C_{\max} =$ **makespan of schedule**

$$
\begin{array}{rll}
\min C_{\max} & & \\
\textbf{s.t.} & & \\
\sum_{i=1}^{m} \frac{x_{ij}}{p_{ij}} = 1 & j = 1, \ldots, n & \textbf{each job runs} \\
\sum_{i=1}^{m} x_{ij} \leq C_{\max} & j = 1, \ldots, n & \textbf{each job;s running time} \\
\sum_{j=1}^{n} x_{ij} \leq C_{\max} & i = 1, \ldots, m & \textbf{each machine load}
\end{array}
$$

An example

	J_1	J_2	J_3	J_4
M_1	**2**	**4**	**3**	**6**
M_2	**6**	**10**	**1**	**4**
M_3	**6**	∞	**3**	**8**

A solution to the linear program (not necessarily optimal)

We can give the solution in a table with entries describing x_{ij}

	J_1	J_2	J_3	J_4
M_1	$\frac{3}{4}$	**4**	**0**	**0**
M_2	**0**	**0**	**1**	$3\frac{1}{2}$
M_3	$3\frac{3}{4}$	**0**	**0**	**1**

$C_{\max} = 4\frac{3}{4}$.

Verification that each job runs:

Job 1: $\frac{3/4}{2} + \frac{15/4}{6} = 1$

Job 2: $\frac{4}{4} = 1$

Job 3: $\frac{1}{1} = 1$

Job 4: $\frac{7/2}{4} + \frac{1}{8} = 1$

This assigns jobs to machines, but doesn't assign the jobs to particular times. To do so we need an algorithm.

Assigning jobs to time slots

We can compute, for each job, and each machine, how much time it needs. In the current example:

	J_1	J_2	J_3	J_4	**total**
M_1	$\frac{3}{4}$	**4**	**0**	**0**	$4\frac{3}{4}$
M_2	**0**	**0**	**1**	$3\frac{1}{2}$	$4\frac{1}{2}$
M_3	$3\frac{3}{4}$	**0**	**0**	**1**	$4\frac{3}{4}$
total	$4\frac{1}{2}$	**4**	**1**	$4\frac{1}{2}$	

	J_1	J_2	J_3	
M_1	2	3	7	12
M_2	0	1	3	4
M_3	4	5	1	11
	6	8	11	

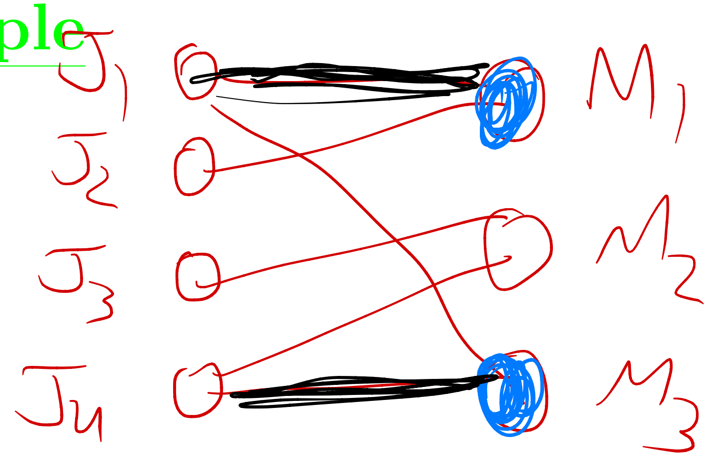
Now, if we are to find a schedule of length $C_{\max} = 4\frac{3}{4}$, it is clear that we ***must*** at all times, keep machines 1 and 3 busy. These machines are considered ***tight***. As we will soon see, it will also be possible to have a tight jobs.

The algorithm

- **We now must choose a subset of jobs to run on machines, so that *each tight job is being run and each tight machine is being used.***
- **We continue running these jobs on these machine until either**
  - **a new machine becomes tight, or**
  - **a new job becomes tight, or**
  - **one of the jobs exhausts its processing time on a machine on which it is running.**
- **To compute the jobs and machines, we solve a matching problem.**
- **After one of the three cases above occurs, we update our processing times to be remaining processing times, and repeat.**

The example

	J_1	J_2	J_3	J_4	**total**
M_1	$\frac{3}{4}$	**4**	**0**	**0**	$4\frac{3}{4}$
M_2	**0**	**0**	**1**	$3\frac{1}{2}$	$4\frac{1}{2}$
M_3	$3\frac{3}{4}$	**0**	**0**	**1**	$4\frac{3}{4}$
total	$4\frac{1}{2}$	**4**	**1**	$4\frac{1}{2}$	

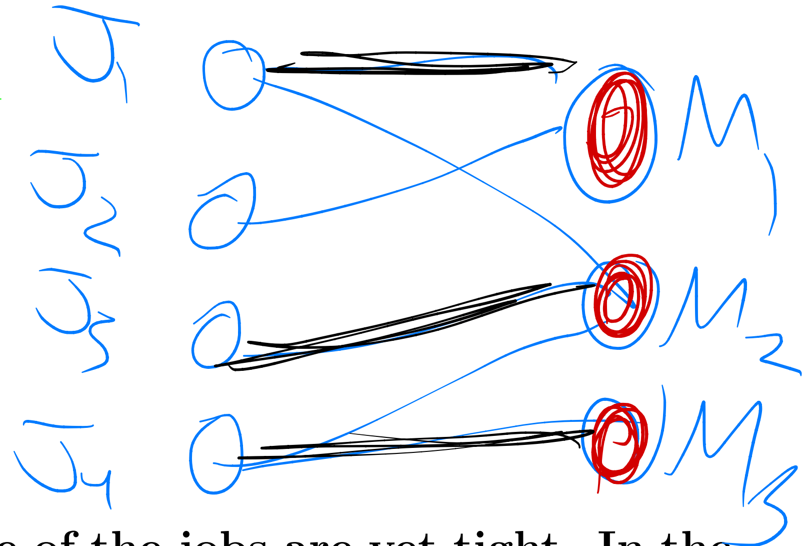


Machine 1 and Machine 3 are tight. Thus we can choose any 2 jobs, as long as one runs on machine 1 and one runs on machine 3. We'll choose J_1 on machine 1 and J_4 on machine 3.

We'll run for $\frac{1}{4}$ units of time, because after $\frac{1}{4}$ units of time, machine 2 and jobs 1 and 4 will become tight. The beginning of the schedule looks like:

We now update the processing times of job 1 on machine 1 and job 4 on machine 3, along with the totals.

Step 2

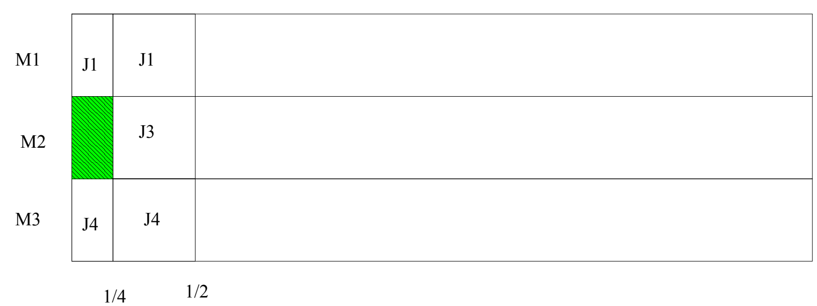


	J_1	J_2	J_3	J_4	total
M_1	$\frac{1}{2}$	**4**	**0**	**0**	$4\frac{1}{2}$
M_2	**0**	**0**	**1**	$3\frac{1}{2}$	$4\frac{1}{2}$
M_3	$3\frac{3}{4}$	**0**	**0**	$\frac{3}{4}$	$4\frac{1}{2}$
Total	$4\frac{1}{4}$	**4**	**1**	$4\frac{1}{4}$	

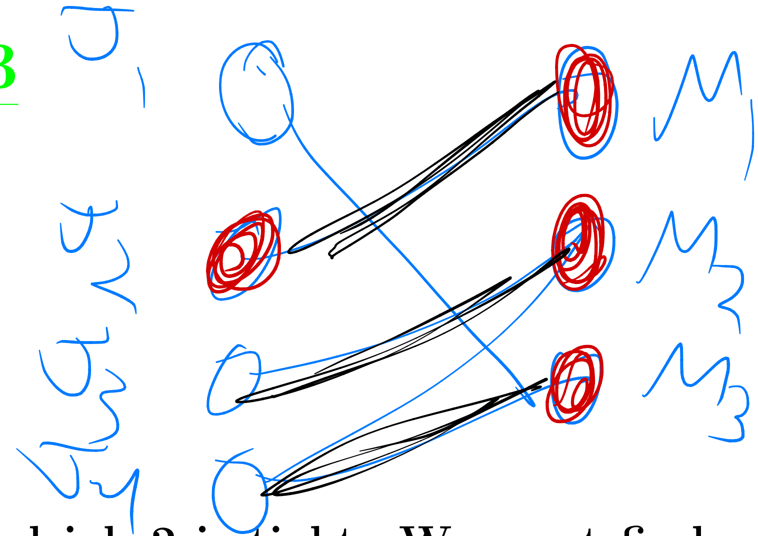
Now all the machines are tight, but none of the jobs are yet tight. In the next time interval, we must keep all machines busy. We do so by finding a matching between jobs and machines. We choose to match:

- **Job 1 to Machine 1**
- **Job 3 to Machine 2**
- **Job 4 to Machine 3**

Note that after $\frac{1}{2}$ units of time job 2 on machine 1 will become tight. So we execute our schedule for $\frac{1}{2}$ time units, and obtain:



Step 3

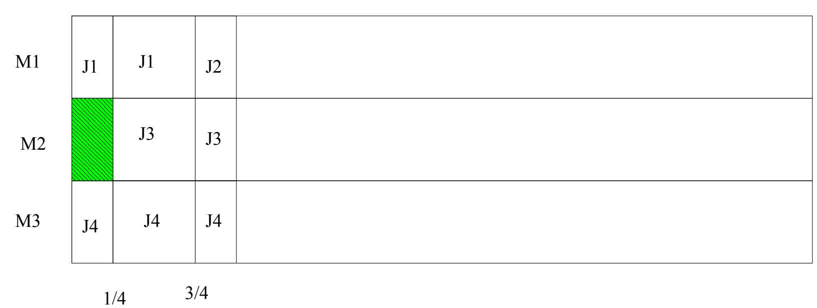


	J_1	J_2	J_3	J_4	total
M_1	**0**	**4**	**0**	**0**	4
M_2	**0**	**0**	$\frac{1}{2}$	$3\frac{1}{2}$	4
M_3	$3\frac{3}{4}$	**0**	**0**	$\frac{1}{4}$	4
Total	$3\frac{3}{4}$	**4**	$\frac{1}{2}$	$3\frac{3}{4}$	

Now all the machines are still tight, and job 2 is tight. We must find a matching that uses all the machines, and job 2. Such a matching is:

- **Job 2 to Machine 1**
- **Job 3 to Machine 2**
- **Job 4 to Machine 3**

Note that after $\frac{1}{4}$ units of time job 1 on machine 3 will become tight and job 4 on machine 4 will terminate. So we execute our schedule for $\frac{1}{4}$ time units, and obtain:



Updating processing times:

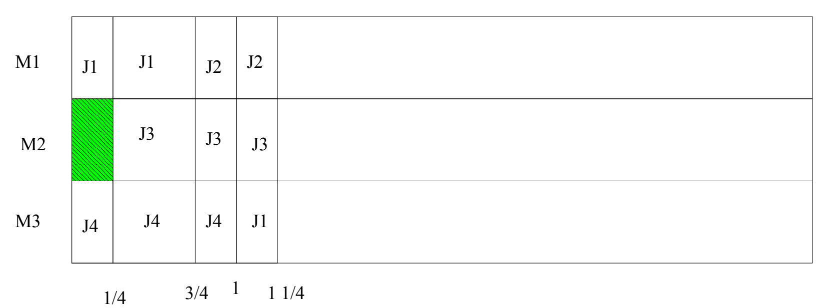
Step 4

	J_1	J_2	J_3	J_4	**total**
M_1	**0**	$3\frac{3}{4}$	**0**	**0**	$3\frac{3}{4}$
M_2	**0**	**0**	$\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$
M_3	$3\frac{3}{4}$	**0**	**0**	**0**	$3\frac{3}{4}$
Total	$3\frac{3}{4}$	$\mathbf{3}\frac{3}{4}$	$\frac{1}{4}$	$3\frac{1}{2}$	

Now all the machines are still tight, and jobs 1 and 2 are tight. We must find a matching that uses all the machines, and jobs 1 and 2. Such a matching is:

- **Job 2 to Machine 1**
- **Job 3 to Machine 2**
- **Job 1 to Machine 3**

Note that after $\frac{1}{4}$ units of time job 4 on machine 2 will become tight So we execute our schedule for $\frac{1}{4}$ time units, and obtain:



Updating processing times:

Step 5

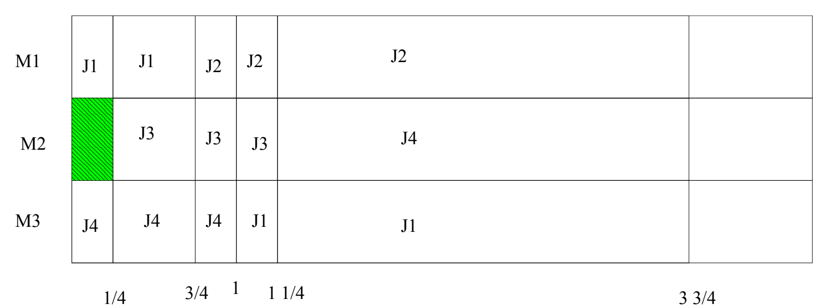
Birkoff-Von Neuman thm. about doubly stochastic matrices

	J_1	J_2	J_3	J_4	total
M_1	$\mathbf{0}$	$3\frac{1}{2}$	$\mathbf{0}$	$\mathbf{0}$	$3\frac{1}{2}$
M_2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$3\frac{1}{2}$	$3\frac{1}{2}$
M_3	$3\frac{1}{2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$3\frac{1}{2}$
Total	$3\frac{1}{2}$	$\mathbf{3}\frac{1}{2}$	0	$3\frac{1}{2}$	

Now all the machines are still tight, and jobs 1, 2 and 4 are tight. We must find a matching that uses all the machines, and jobs 1, 2 and 4. Such a matching is:

- **Job 2 to Machine 1**
- **Job 4 to Machine 2**
- **Job 1 to Machine 3**

These can all run to completion, and we obtain the final schedule:



This procedure will always find a valid schedule.

Step 5

	J_1	J_2	J_3	J_4	total
M_1	**0**	$3\frac{1}{2}$	**0**	**0**	$3\frac{1}{2}$
M_2	**0**	**0**	**0**	$3\frac{1}{2}$	$3\frac{1}{2}$
M_3	$3\frac{1}{2}$	**0**	**0**	**0**	$3\frac{1}{2}$
Total	$3\frac{1}{2}$	$\mathbf{3}\frac{1}{2}$	0	$3\frac{1}{2}$	

Now all the machines are still tight, and jobs 1, 2 and 4 are tight. We must find a matching that uses all the machines, and jobs 1, 2 and 4. Such a matching is:

- Job 2 to Machine 1
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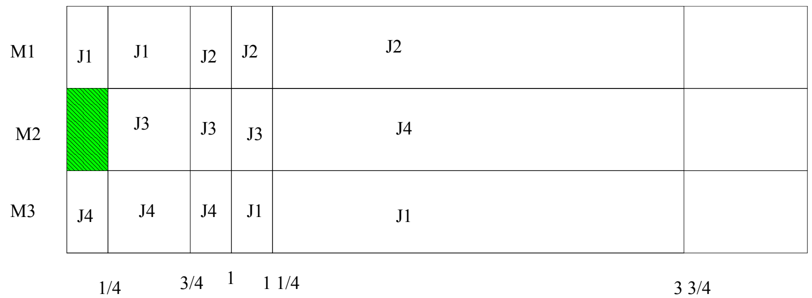
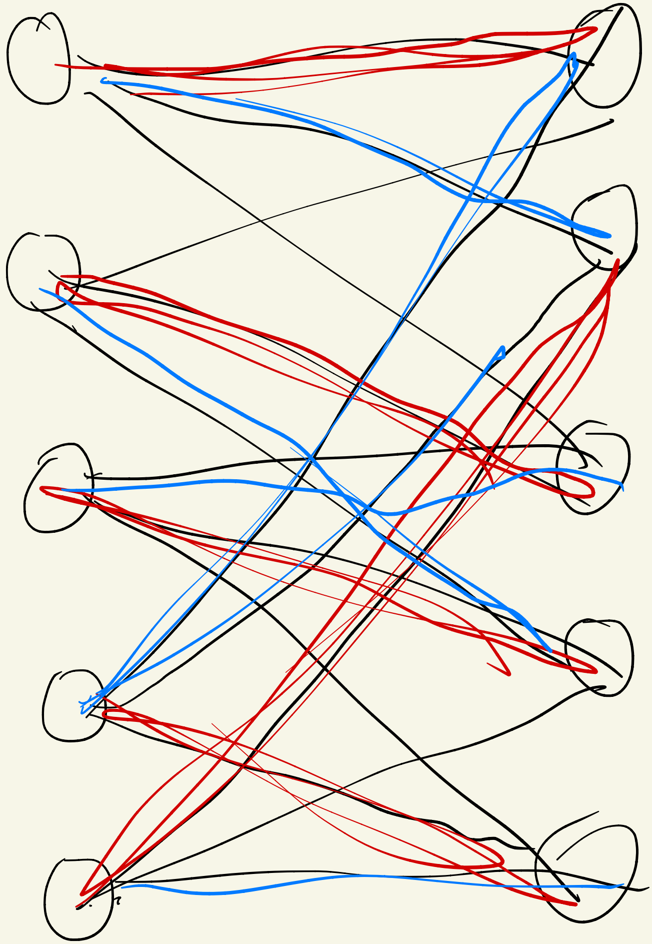
These can all run to completion, and we obtain the final schedule:



This procedure will always find a valid schedule.

Related Fact



Every node has degree 3

Always a way to decompose the graph into 3 matchings

"Dance Hall Thm"